

# On the Decomposition of Spinor Fields which satisfy a Nonlinear Higher Order Equation

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Z. Naturforsch. **38a**, 1293–1295 (1983); received July 8, 1983

A field theory which is based entirely on fermion fields is non-renormalizable if the kinetic energy contains only derivatives of first order and therefore higher derivatives have to be included. Such field theories may be useful for describing preons and their interaction. In this note we show that a spinor field which satisfies a higher order field equation with an arbitrary nonlinear selfinteraction can be written as a sum of fields which satisfy first order equations.

## Introduction

One way of trying to solve the outstanding problems of the standard model consists in considering models in which leptons and quarks are composite particles built up out of even smaller constituents which are (sometimes) called preons. This idea entails two basic questions: what sort of particles are these preons and how do they interact? Certainly some of them must have spin  $\frac{1}{2}$ . For aesthetical and economical reasons we would like to assume that they all have spin  $\frac{1}{2}$ . In analogy to the standard model their interaction is usually assumed to be based on a non-abelian gauge symmetry. Apart from the fact that in this way elementary particles with spin 1 are introduced in addition to the spin  $\frac{1}{2}$  preons there is another reason why nature may not repeat itself on this more fundamental level. Non-abelian gauge theories are asymptotically free, which means that the interaction becomes small if the preons are close together. On the other hand, leptons and quarks are very small, smaller than  $10^{-16}$  cm. Therefore, due to the uncertainty principle the kinetic energy of the preons must be higher than 100 GeV. The mass of the composite systems, e.g. the electron, is very small on that scale. Hence one needs a sufficiently high binding energy and it is not at all clear whether an asymptotically free theory can provide such a binding energy. This problem does not arise in QCD because hadrons have an extension of  $10^{-13}$  cm, so that the kinetic energy of quarks is of the same order of magnitude as the mass of the

hadrons. Hence the asymptotic freedom of QCD not only poses no problem but fits in rather nicely. On the preon level, however, asymptotic freedom creates a problem. We would therefore prefer to introduce a selfinteraction of the preon fields because such an interaction is especially effective if the particles are close together and because in this case there is no need for elementary spin 1 bosons.

Selfinteraction of fermion fields, however, poses another problem. Since the interaction must be Lorentz invariant it has to be a product of at least four fermion fields. This leads to a non-renormalizable theory if the kinetic term of the Lagrangian is of first order in the derivatives. If one insists on renormalizability one has to include higher order derivatives. Such theories after quantization lead to negative-norm states in Fock space and hence to a non-unitary S-matrix at tree level. The history of higher order field theories and the problem of indefinite metric have been discussed in a recent publication [1]. Stumpf [2] has recently shown that in such theories for high preon masses bound states with positive norm and low mass exist.

Let us therefore consider a theory which contains a finite set of Dirac spinors  $\psi_k$  ( $k = 1, \dots, N$ ) satisfying a nonlinear higher order equation. We will show in this note that such fields can be decomposed into fields  $\phi_{ki}$  which satisfy nonlinear first order equations. Such a decomposition has two advantages. If we switch off the interaction the  $\phi_{ki}$  satisfy the free Dirac equation and hence there is a particle interpretation. In addition the commutator of the Hamiltonian with a field variable yields essentially the time derivative of that field variable. If the equation of motion is of first order then one can by eliminating the time derivative equations for the

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eigenvalues and eigenstates of the Hamiltonian which are of the same type as the non-relativistic Schrödinger equation.

This decomposition of fields which satisfy higher order field equations was introduced by Pais and Uhlenbeck [3] for free fields. When they considered the interaction of a spinor field with the electromagnetic field then they encountered difficulties. Stumpf [4] has shown recently that if a spinor field satisfies a nonlinear higher order equation then the decomposition leads to fields which obey a nonlinear first order equation. The equivalence of these two equations, however, was only shown for bound state solutions. In this note we will prove the same theorem without using this assumption.

### The decomposition

In the following we will only treat the case  $N = 1$ , so that we have only one spinor field  $\psi_1 = \psi$ . All considerations are easily generalized to the case  $N > 1$ . Let us assume that we have a nonlinear higher order equation

$$\prod_{i=1}^n (i \gamma^\mu \partial_\mu - m_i) \psi = V(\psi), \quad (1)$$

where

$$m_i \neq m_k \quad \text{if} \quad i \neq k, \quad (2)$$

and  $V$  is an arbitrary function, and let  $M_\psi$  denote the set of all solutions of this equation. Starting from a solution  $\psi$  of (1) we can define the functions

$$\varphi_i := \eta_i \prod_{\substack{k=1 \\ k \neq i}}^n (i \gamma^\mu \partial_\mu - m_k) \psi, \quad (3)$$

where

$$\eta_i := \left( \prod_{\substack{k=1 \\ k \neq i}}^n (m_i - m_k) \right)^{-1}. \quad (4)$$

Let us further consider the following set of first order nonlinear equations:

$$(i \gamma^\mu \partial_\mu - m_i) \varphi_i = \eta_i V \left( \sum_{j=1}^n \varphi_j \right), \quad i = 1, \dots, n, \quad (5)$$

and let  $M_\varphi$  denote the set of all solutions of this equation. The content of this note will then be the proof of the following theorem:

The map  $F$  which is given by (3) is one-to-one and maps  $M_\psi$  onto  $M_\varphi$ . The inverse of this map is given by

$$\psi = \prod_{i=1}^n \varphi_i. \quad (6)$$

Let first derive the identity

$$\sum_{i=1}^n \eta_i \prod_{\substack{k=1 \\ k \neq i}}^n (\alpha - m_k) = 1 \quad (7)$$

for arbitrary  $\alpha$ . If we define

$$f(\alpha) = \prod_{i=1}^n (\alpha - m_i) \quad (8)$$

then because of (2) one has

$$\frac{1}{f(\alpha)} = \sum_{j=1}^n \frac{A_j}{\alpha - m_j}. \quad (9)$$

To determine the coefficients  $A_j$  we multiply both sides of this equation with  $f(\alpha) (\alpha - m_r)$ . This yields

$$\alpha - m_r = \sum_{\substack{j=1 \\ j \neq r}}^n \frac{A_j}{\alpha - m_j} (\alpha - m_r) f(\alpha) + A_r f(\alpha). \quad (10)$$

If we differentiate both sides of this equation with respect to  $\alpha$  and then put  $\alpha = m_r$  then we obtain

$$1 = A_r f'(m_r). \quad (11)$$

From the definition of  $f(\alpha)$  (Eq. (8)) it follows that

$$f'(m_r) = \prod_{\substack{i=1 \\ i \neq r}}^n (m_r - m_i). \quad (12)$$

Equations (4), (11) and (12) imply

$$A_r = \eta_r. \quad (13)$$

Inserting this equation into (9) and multiplying the resulting equation with  $f(\alpha)$  one obtains the wanted identity (7).

Equation (7) is an identity in  $\alpha$ , i.e. it is true for arbitrary  $\alpha$ . Therefore, if we write the right-hand side of (7) as a polynomial in  $\alpha$  then the constant term equals 1 and the coefficients of all higher powers of  $\alpha$  are zero. That means that (7) remains true if we choose  $\alpha$  to be an operator, e.g.  $\alpha = i \gamma^\mu \partial_\mu$ .

To prove our theorem we first show that the map  $F$  given by (3) maps  $M_\psi$  into  $M_\varphi$ . That means we show that if  $\psi$  is a solution of (1) then the  $\varphi_i$  defined

by (3) are solutions of (5). Now from the definition (3) and identity (7) for  $\alpha = i \gamma^\mu \partial_\mu$  it follows that

$$\sum_{i=1}^n \varphi_i = \sum_{i=1}^n \eta_i \prod_{\substack{k=1 \\ k \neq i}}^n (i \gamma^\mu \partial_\mu - m_k) \psi = \psi. \quad (14)$$

Therefore, one has from (3) and (1)

$$\begin{aligned} (i \gamma^\mu \partial_\mu - m_i) \varphi_i &= \eta_i \prod_{k=1}^n (i \gamma^\mu \partial_\mu - m_k) \psi \\ &= \eta_i V(\psi) = \eta_i V\left(\sum_{j=1}^n \varphi_j\right) \end{aligned} \quad (15)$$

which is (5).

To prove the remaining assertions of the theorem we consider the map

$$\bar{F}(\varphi_1, \dots, \varphi_n) = \sum_{i=1}^n \varphi_i \quad (16)$$

defined on  $M_\varphi$ . From (5) and (7) we find

$$\begin{aligned} \prod_{i=1}^n (i \gamma^\mu \partial_\mu - m_i) \bar{F}(\varphi_1, \dots, \varphi_n) \\ &= \prod_{i=1}^n (i \gamma^\mu \partial_\mu - m_i) \sum_{j=1}^n \varphi_j \\ &= \sum_{j=1}^n \prod_{\substack{i=1 \\ i \neq j}}^n (i \gamma^\mu \partial_\mu - m_i) \eta_j V\left(\sum_{k=1}^n \varphi_k\right) \\ &= V(\bar{F}(\varphi_1, \dots, \varphi_n)). \end{aligned} \quad (17)$$

Therefore,  $\bar{F}(\varphi_1, \dots, \varphi_n)$  is a solution of (1) and hence  $\bar{F}$  maps  $M_\varphi$  into  $M_\psi$ .

We will now show that

$$\bar{F}F = 1_{M_\psi}, \quad (18)$$

$$F\bar{F} = 1_{M_\varphi}, \quad (19)$$

where  $1_{M_\psi}$  and  $1_{M_\varphi}$  are the identity map on  $M_\psi$  and  $M_\varphi$ , respectively.

It then follows from (18) that  $F$  is one-to-one and from (19) that  $F$  maps  $M_\psi$  onto  $M_\varphi$ . In addition it

follows that  $\bar{F} = F^{-1}$  and hence the proof of the theorem is complete.

From (16), (3), and (7) one has

$$\bar{F}F\psi = \sum_{i=1}^n \eta_i \prod_{\substack{k=1 \\ k \neq i}}^n (i \gamma^\mu \partial_\mu - m_k) \psi = \psi \quad (20)$$

and hence (18) is true. To prove (19) we have to show that

$$(F\bar{F}(\varphi_1, \dots, \varphi_n))_i = \varphi_i \quad (21)$$

if  $(\varphi_1, \dots, \varphi_n) \in M_\varphi$ . For simplicity we show this only for  $i = 1$ . From (3) and (16) one finds

$$(F\bar{F}(\varphi_1, \dots, \varphi_n))_1 = \eta_1 \prod_{k=2}^n (i \gamma^\mu \partial_\mu - m_k) \sum_{j=1}^n \varphi_j. \quad (22)$$

Since  $(\varphi_1, \dots, \varphi_n) \in M_\varphi$  it follows from (5) that

$$\frac{1}{\eta_j} (i \gamma^\mu \partial_\mu - m_j) \varphi_j = \frac{1}{\eta_1} (i \gamma^\mu \partial_\mu - m_1) \varphi_1. \quad (23)$$

From this it follows that

$$\begin{aligned} (F\bar{F}(\varphi_1, \dots, \varphi_n))_1 \\ &= \eta_1 \prod_{k=2}^n (i \gamma^\mu \partial_\mu - m_k) \varphi_1 \\ &\quad + \sum_{j=2}^n \eta_1 \prod_{k=2}^n (i \gamma^\mu \partial_\mu - m_k) \varphi_j \\ &= \eta_1 \prod_{k=2}^n (i \gamma^\mu \partial_\mu - m_k) \varphi_1 \\ &\quad + \sum_{j=2}^n \eta_1 \prod_{\substack{k=2 \\ k \neq j}}^n (i \gamma^\mu \partial_\mu - m_k) \frac{\eta_j}{\eta_1} (i \gamma^\mu \partial_\mu - m_1) \varphi_1 \\ &= \sum_{j=1}^n \eta_j \prod_{\substack{k=1 \\ k \neq j}}^n (i \gamma^\mu \partial_\mu - m_k) \varphi_1 = \varphi_1 \end{aligned}$$

and hence (19) is true.

I am grateful to the Deutsche Forschungsgemeinschaft for financial support.

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